Introduction to scaling concepts in disordered magnets

Types of disorder :

Consider an Ising magnet described by the following Hamiltonian :

$$H = - \sum_{r, r'} J(r, r') \, s(r) \, s(r') - \sum_{r} h(r) \, s(r)$$

where $\{J(r,r')\}$ and $\{h(r)\}$ are static, random variables. This kind of disorder is called quenched disorder. On the other hand, if $\{J(r,r')\}$ and $\{h(r)\}$ are themselves dynamical variables that are summed over in the partition f^n, the disorder is called annealed. We will mostly concern ourselves with the quenched case.

Randomness in field ($\{h(r)\}$) vs randomness in coupling strength ($\{J(r,r')\}$)

These two distinct kinds of disorders have qualitatively different effects. As one may guess, random field tends to have more dramatic effect on the phase diagram. Lets consider their effects at a qualitative level.

Random fields and stability of spontaneous symmetry breaking (Imry-Ma criterion)

Consider an Ising magnet with the Hamiltonian

$$H = -J \sum_{\langle r r' \rangle} s(r)\, s(r') - \sum_{r} h(r)\, s(r)$$

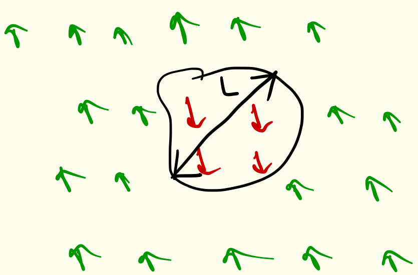
where $\{h(r)\}$ are uncorrelated random variables with zero mean and finite variance:

$$\langle h(r) \rangle = 0$$

$$\langle h(r)\, h(r') \rangle = W\, \delta(r-r')$$

The random field breaks the Ising symmetry locally. However, in a statistical sense, the Ising sym is still preserved since $\langle h(r) \rangle = 0$. Therefore, it's a legitimate question whether there is a possibility of long-range order where magnetization is either positive or negative everywhere. We will now outline an argument by Imry and Ma that suggests an answer to this question.

Consider a putative SSB phase where spins point all up except a puddle of down-spins in a region where $h(r) < 0$ (so that locally h favors down spins). The question is whether



such a domain of down spins is stable. If yes, then SSB cannot occur. The competition here is between the following two factors:

(i) The domain costs an energy $\sim \sigma L^{d-1}$ where σ is the domain wall surface tension. L is the domain's linear length.

(ii) Due to random fields, the domain wall gains energy of order

$$\sqrt{\left\langle \left(\sum_{r \in \text{domain wall}} h(r) \right)^2 \right\rangle} \approx \sqrt{\langle h^2(r) \rangle L^d}$$

$$= \sqrt{W} L^{d/2}.$$

Essentially one is summing L^d random numbers, and due to the central limit theorem, the magnitude is of the order of $L^{d/2}$.

Therefore, if $\sigma L^{d-1} > \sqrt{W} L^{d/2}$ as $L \to \infty$, then domain wall will shrink since it is unfavorable.

$\Rightarrow$ In $d > 2$ domain walls are unstable and one expects that SSB is stable.

And when $d < 2$ one expects that the system will break into numerous domains and SSB does not occur. The size of the domains, L_{dw}, may be estimated by comparing the two competing terms:

$$\sigma L_{dw}^{d-1} \sim \sqrt{W} \, L_{dw}^{d/2}$$

$$\Rightarrow \qquad L_{dw} \sim \left(\frac{W}{\sigma^2}\right)^{1/d-2}$$

Rigorous calculations show that SSB does not occur in $d \leq 2$. One finds that in $d=2$, the domain wall size is exponentially large in $\frac{1}{W}$: $L_{dw} \sim e^{1/W}$.

Generalization to continuous sym breaking:

Next consider a system with continuous sym, e.g. O(N) model, in the presence of a random field:

$$H = -J \sum_{\langle r r' \rangle} n(r) \cdot n(r') - \sum_r \vec{h}(r) \cdot \vec{n}(r)$$

Again $\langle \vec{h}(r) \rangle = 0 \quad \langle \vec{h}(r) \cdot \vec{h}(r') \rangle = W \delta(\vec{r} - \vec{r}')$.

The key difference from the case of discrete symmetry is that now the spins can rotate continuously and therefore the domain-wall cost is lowered:

$$\text{energy cost} \sim \rho \int_{x \in \text{domain wall}} d^d x \, (\nabla \vec{n})^2$$

$$\sim \rho \frac{L^d}{L^2} \sim \rho L^{d-2}.$$

The estimate for the energy gain due to random fields is unchanged since that didn't rely on whether the symmetry is continuous or discrete. Therefore, energy gain $\sim \sqrt{W} L^{d/2}$

For SSB to be stable:

$$\rho L^{d-2} > \sqrt{W} L^{d/2}$$

$\Rightarrow$ SSB stable only in $d > 4$.

Random couplings and stability of phase transitions

Instability of first-order transitions against random T_c :

Consider a system which has locally random couplings so that the effective T_c fluctuates as a f^n of the location in the system. If the clean system (i.e. without disorder) undergoes a first-order transition, one may ask if the random system also undergoes a first-order transition.

A first order transition is characterized by macroscopic co-existence of two distinct states e.g. liquid and gas across the boiling transition.

Random-T_c will favor one state over the other depending on the location. Consider a domain of one state that is locally favored while being embedded in a bigger region of the other state.

If this domain survives then the phase coexistence is no longer possible since the system will break-up into innumerous domains dictated by local T_c. The energy balance is essentially same as the one discussed above in the context of the stability of SSB.

Free energy cost of a domain $\sim \sigma L^{d-1}$

Energy gain due to domain $\sim \sqrt{W}\, L^{d/2}$ where W = variance of T_c.

$\Rightarrow$ When $d \leq 2$ domains are favored and macroscopic phase coexistence is impossible

$\Rightarrow$ in $d \leq 2$, first-order transitions do not occur in the presence of random T_c disorder. Typically, in such scenarios, a first order transition gets rounded and gives way to a continuous transition.

Random couplings and Stability of critical points (Harris Criterion)

One might naively think that a second-order transition may always get smeared out by random variation of T_c, with different regions undergoing transitions at different temperatures. This however is not the case. Let's divide the system into blocks of size ξ, the correlation length. Each such block will have its own T_c, which we can denote as $T_c(i)$ where i denotes the block number. If the variance of $T_c(i)$ over i is bigger than $T-T_c$, where T_c is the global T_c in the absence of disorder, then a uniform transition governed by the clean critical point is impossible. However, if variance $<|T-T_c|$, then the system is approximately uniform and one expects that the long distance physics is governed by the clean RG fixed point.

Therefore, let's estimate the variance of $T_c(i)$.

$$\# \text{ of blocks} = \left(\frac{L}{\xi}\right)^d$$

$\Rightarrow$ variance in $T_c \sim [\# \text{ of blocks}]^{1/2}$ via the central limit theorem.

$$\Rightarrow \quad \Delta T_c \sim \xi^{-d/2}.$$

On the other hand, the distance $|T - T_c|$ from the clean critical T_c, $|T - T_c| \sim \xi^{-1/\nu}$.

For the fluctuations in T_c to be unimportant,

$$\Delta T_c \ll |T - T_c|$$

$$\Rightarrow \quad \xi^{d/2} \gg \xi^{1/\nu}$$

$$\Rightarrow \quad \frac{d}{2} > \frac{1}{\nu} \quad \text{or} \quad \boxed{d\nu > 2}$$

When this inequality is satisfied, then the system appears asymptotically clean at the longest distances and therefore, the critical exponents are determined by the clean fixed point.

Renormalization Group approach to obtain Harris Criterion

Consider a system with disorder so that for given realization of the disorder $\{m\}$, the partition fn is given by,

$$Z(m) = \mathrm{Tr}_s \, e^{-H(m,s)}$$

where $\{s\}$ are the underlying d.o.f. (e.g. Ising spins). To obtain physical quantities such as the free energy, it is crucial to take average over $\{m\}$ of extensive quantities, such as the free energy itself. This is because non-extensive quantities such as Z, are likely to be not self-averaging. Therefore, one needs,

$$\overline{F} = \mathrm{Tr}_m \, P(m) \, F(m)$$

where $F(m) = -\log Z(m)$ and $P(m)$ is the p.d.f. for the randomness $\{m\}$.

This quantity is expected to be self-averaging and its fluctuations are expected to be $O(1/\sqrt{V})$ where V is the volume. Similarly, spatially averaged correlation f^{n}s

$$\overline{\left\langle \sum_r s(r_1+r)\, s(r_2+r) \right\rangle}$$

are also expected to be self-averaging where $\langle\ \rangle$ denotes expectation value w.r.t. to a single disorder realization and $\overline{}$ denotes average of the resulting quantity over all disorder configs.

Explicitly, $\overline{\left\langle \sum_r s(r_1+r)\, s(r_2+r) \right\rangle}$

$$= \mathrm{Tr}_m \left[\frac{\mathrm{Tr}_s \sum_r s(r_1+r)\, s(r_2+r)\, e^{-H(m,s)}}{\mathrm{Tr}_s\, e^{-H(m,s)}} \right] P(m)$$

Note that this does not equal

$$\frac{\mathrm{Tr}_m\, \mathrm{Tr}_s \sum_r s(r_1+r)\, s(r_2+r)\, e^{-H(m,s)}\, P(m)}{\mathrm{Tr}_m\, \mathrm{Tr}_s\, e^{-H(m,s)}\, P(m)}.$$

One may verify that the correct expression can be obtained from $\overline{F}$ by taking suitable derivatives.

The hard part of this problem is that averaging of $\log Z_1$, which enters the expression for $\overline{F}$, is rather intractable due to log not being a polynomial.

One method that has been developed to tackle this issue is the Replica trick, which exploits the fact that

$$\log Z = \underset{x \to 0}{\mathrm{Lt}} \frac{Z^x - 1}{x}$$

Since one is taking $x \to 0$, x is necessarily a real number. However, to make the problem tractable, one considers x to be an integer, so that various averages over the disorder can be calculated and then analytically continue to n being a real number and finally take the limit. One issue with this approach is that the limits $V \to \infty$ and $n \to 0$ may not commute. Physically, one needs to take the limit $n \to 0$ before $V \to \infty$, so that one obtains $\lim_{V \to \infty} \log Z$. However, in practice, its sometimes only possible to take $V \to \infty$ before $n \to 0$ which can give incorrect results. An example is provided by the spin-glass problem which we may discuss a bit, time permitting. For now, lets derive the Harris criteria using replica trick.

Consider a clean system at an RG fixed point that describes a second-order phase transition (e.g. the Wilson-Fisher fixed point for an $O(n)$ model). Lets perturb this fixed point with a term that describes randomness in energy-density:

$$H = \underbrace{H_0}_{\text{Clean fixed-point}} + \sum_r \underbrace{E(r)}_{\text{energy density.}} \underbrace{m(r)}_{\text{random variable.}}$$

For example, if

$$\mathcal{H} = \int d^d r \left[(\nabla \varphi)^2 + [t + \delta t(r)] \varphi^2(r) + u\, \varphi^4(r) \right]$$

then $m^2(r) = \delta t(r)$.

We are interested in calculating $\overline{F} = \overline{\log Z}$ $= \lim_{n \to 0} \frac{\overline{Z^n} - 1}{n}$. Therefore, we need to calculate

$\overline{Z^n}$:

$$\overline{Z^n} = \mathrm{Tr}_m \left[\mathrm{Tr}_s\, e^{-\left[H_0 + \sum_r E(r)\, m(r)\right]} \right]^m P(m)$$

$$= \mathrm{Tr}_m \prod_{a=1}^{m} \mathrm{Tr}_{s_a}\, e^{-\left[H_0(s_a, m) + \sum_r E(s_a, r)\, m(r)\right]} P(m)$$

$$= \mathrm{Tr}_m\, \mathrm{Tr}_{s_a}\, e^{-\sum_a H_0^a - \sum_a \sum_r m(r)\, E^a(r)} P(m)$$

where $H_0^a \equiv H_0(\{S^a\})$ and $E^a(r) \equiv E(S_a, r)$

$$= \mathrm{Tr}_{S_a}\, e^{-\sum_a H_0^a} \; \mathrm{Tr}_m\, P(m)\, e^{-\sum_a \sum_r E^a(r)\, m(r)}$$

Assuming weak-disorder, one may evaluate the averaging over m perturbatively using the cumulant expansion : $\overline{e^{-x}} = e^{-[\overline{x} - [\frac{\overline{x^2} - \overline{x}^2}{2}] + \dots]}$

$$\Rightarrow \quad \overline{\mathrm{Tr}_m\, P(m)\, e^{-\sum_a \sum_r E^a(r)\, m(r)}}$$

$$\exp\left[-\overline{m} \sum_a \sum_r E^a(r) + \frac{1}{2}\left\{ \overline{\left(\sum_{ra} E^a(r)\, m(r)\right)^2} - \left[\overline{\sum_{ra} E^a(r)\, m(r)}\right]^2 \right\}\right]$$

$$= \exp\left[-\overline{m} \sum_a \sum_r E^a(r) + \frac{1}{2} \sum_{rr'} E^a(r)\, E^b(r') \left[\overline{m(r)\, m(r')} - \overline{m}^2\right]\right]$$

The first term, $-\overline{m} \sum_r \sum_a E^a(r)$ is replica-diagonal and therefore, when added to $\sum_a H_0^a$, simply shifts the T_c. The second term can be simplified using OPE:

$$E^a(r)\, E^b(r') \sim E^a(r)\, \delta_{ab} + (1-\delta_{ab})\, E^a(r)\, E^b(r) + \text{higher derivative terms.}$$

Again, the first term in this expression simply shifts the T_c. The second term is more interesting. If it is irrelevant, then $\overline{Z^n}$ is simply given by $(\overline{Z})^n$ i.e. disorder is irrelevant. Therefore, all we need to do to figure out the relevance / irrelevance of disorder is to calculate the RG eigenvalue of this term.

$$\left\langle \sum_{a\neq b} E^a(r)\, E^b(r) \sum_{a'\neq b'} E^{a'}(r')\, E^{b'}(r') \right\rangle$$

$$\sim \sum_{a\neq b} \langle E^a(r)\, E^a(r')\rangle \langle E^b(r)\, E^b(r')\rangle$$

$$\sim 4\, {}^nC_2\, \frac{1}{|r-r'|^{4x_E}}$$ where $x_E = d - y_t$ is the RG eigenvalue of E.

$\Rightarrow$ RG eigenvalue of this term

$$= d - 2x_E = d - 2\left(d - \frac{1}{\nu}\right)$$

$$= \frac{2}{\nu} - d$$ where we have used

$y_t = 1/\nu$.

$\Rightarrow$ when $d > \frac{2}{\nu}$, disorder is perturbatively irrelevant. Thus, we recover the Harris criteria we earlier argued on heuristic grounds.

Perturbatively accessible disordered fixed point

Above we argued that when $d\nu > 2$ i.e. exponent $\alpha = 2 - d\nu < 0$, disorder is perturbatively irrelevant. One may wonder if α is positive and small, perhaps one will flow to a new fixed point that is accessible perturbatively. For 3d Ising, $\alpha \approx 0.11$, therefore such an approach is applicable there. Further, $\alpha = 0$ in $d = 2$ (Onsager's solution). Therefore, it's a reasonable assumption that in $d = 2 + \varepsilon$, $\alpha = O(\varepsilon)$.

As discussed above, the effective Hamiltonian after replica trick is given by,

$$H_{eff} = \sum_a H_0^a - \Delta \sum_r \sum_{a \neq b} \varepsilon_a(r)\, \varepsilon_b(r)$$

where $\Delta = \overline{m^2} - \overline{m}^2$. One may now find the RG flow of Δ using OPE. Note the negative sign in front of the Δ term.

Above, we found that the leading term is proportional to $2 - d_\nu = \alpha$ i.e.

$$\frac{d\Delta}{dl} = \alpha \Delta + \ldots$$

We now calculate the ... using OPE. Since in $\sum_a H^0_a$, replicas are decoupled

$\mathcal{E}_a(r)\,\mathcal{E}_b(r) \propto \delta_{ab}$. More specifically,

$$\mathcal{E}_a \mathcal{E}_b \sim \delta_{ab} + c\, \mathcal{E}_a \delta_{ab},$$

where 'c' is a universal OPE coefficient.

As we earlier discussed, $c = 0$ for $d = 2$ Ising due to the Ising duality which maps $\mathcal{E} \to -\mathcal{E}$. Therefore, its reasonable to assume that in $d = 2+\varepsilon$, $c = O(\varepsilon)$ at most.

To calculate the RG flow of Δ we need,

$$\sum_{a \neq b} \mathcal{E}_a \cdot \mathcal{E}_b \sum_{c \neq d} \mathcal{E}_c \cdot \mathcal{E}_d$$

$$= \left[4(n-2) + 2c^2 \right] \sum_{a \neq b} \mathcal{E}_a \mathcal{E}_b$$

$$\Rightarrow \quad \frac{d(-\Delta)}{dl} = \alpha(-\Delta) - [4(n-2)+2c^2](-\Delta)^2$$

$$\Rightarrow \quad \frac{d\Delta}{dl} = \alpha\Delta + [4(n-2)+2c^2]\,\Delta^2$$

Since $\alpha = O(\varepsilon)$, $c = O(\varepsilon)$, one may drop the c^2 term at $O(\varepsilon)$. Taking $n \to 0$ limit (!), one finds a stable fixed point at $\Delta^* = \frac{\alpha}{8} = O(\varepsilon)$.

To find the exponent ν, we need RG flow for t,

$$\frac{dt}{dl} = y_t^0\, t + 2C_{tOt}\, t\,\Delta$$

where $O = \sum_{a\neq b} \varepsilon_a \varepsilon_b$ and y_t^0 is the RG eigenvalue at the clean fixed point.
Thus, we need the OPE $\sum_{a\neq b} \varepsilon_a \varepsilon_b \sum_c \varepsilon_c$
Plugging in the above expression, one finds

$$\sum_{a\neq b} \varepsilon_a \varepsilon_b \sum_c \varepsilon_c = 2(n-1)\sum_c \varepsilon_c + \ldots$$

$$\Rightarrow \quad \frac{dt}{dl} = y^0{}_t \, t + 4(n-1)\Delta t$$

$$y^0{}_t = \nu_{clean}^{-1} = \frac{d}{2-\alpha_{clean}} \approx \frac{2+\varepsilon}{2-\alpha}$$

$$\approx \quad 1 + \frac{\varepsilon}{2} + \frac{\alpha}{2}.$$ Setting $n = 0$,

$$\Rightarrow \quad y_t = y^0{}_t - 4\Delta^*$$

$$\approx 1 + \frac{\varepsilon}{2} + \frac{\alpha}{2} - \frac{\alpha}{2} \approx 1 + \frac{\varepsilon}{2}$$

$$\Rightarrow \quad \nu_{disorder} = y_t^{-1} \approx 1 - \frac{\varepsilon}{2}.$$

$$\alpha_{disorder} = 2 - d\nu_{disorder}$$

$$= 2 - (2+\varepsilon)(1 - \tfrac{\varepsilon}{2})$$

$$= 0 + O(\varepsilon^2)$$

There's a rigorous theorem due to Chayes et al that at the disordered fixed point, $\alpha < 0$.